Mean Value Theorems in Higher Dimensions and Their Applications

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Abstract

In this paper, we describe the Mean Value Theorem (MVT) and Cauchy Mean Value Theorem (CMVT) when considering an $\mathbb{R}^{n-1}$ dimensional *hyperplane intersects an* $\mathbb{R}^{n-1}$ dimensional *smooth surface in* $\mathbb{R}^n$. *We demonstrate how we derive the the proofs of MVT and CMVT by applying techniques described in [4]. We further discuss how the theorems can be extended by replacing the hyperplane with another smooth surface. Next, we link MVT to problems of finding the extreme values for a smooth function subject to several constraints. We use technological tools to show how we can obtain the solutions that are guaranteed by our theories.*

1 Introduction

Throughout this paper, we assume the Rolle's theorem on a function f, that is differentiable on (a, b) and continuous on $[a, b]$. We recall from [4] that if we consider the parametric curve $r(t) = [g(t), f(t)]$ and the line segment connecting the points $P = (g(a), f(a))$ and $Q = (g(b), f(b))$ intersects $r(t)$, then there exists a $t \in (a, b)$ such that the slope of the secant line PQ is the same as that of tangent line at a point of $r(t)$. This is exactly what the Cauchy Mean Value Theorem states below:

Suppose the function $f : [a, b] \to \mathbb{R}$ *and* $g : [a, b] \to \mathbb{R}$ *are continuous and that their restrictions to* (a, b) *are differentiable. Moreover, assume that* $g'(t) \neq 0$ *for all* t *in* (a, b). *Then there is a point* t *in* (a, b) *at which*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(t)}{g'(t).}$$

Moreover, we note that the proof in [4] suggests that if we consider the line equation connecting PQ as

$$y(t) = m \cdot x(t) + b = \left(\frac{f(b) - f(a)}{g(b) - g(a)} \right) \cdot g(t) + b,$$

where b is the $y-$intercept of the line PQ, and consider the new parametric curve

$$\begin{aligned} r^*(t) &= [g(t), f(t) - y(t)] \\ &= \left[g(t), f(t) - \left(\left(\frac{f(b)-f(a)}{g(b)-g(a)}\right)\cdot g(t) + b\right)\right], \end{aligned}$$

the result of CMVT follows immediately by applying the Rolle Theorem on $r^*(t)$. In many textbooks, for example ([1], page 368) suggests that we may think of MVT in higher dimension as a local behavior involving the directional derivative at one point in a given direction. More specifically, we have the following:

Let U be an open subset of $\mathbb{R}^n$ and suppose the function $f : U \to \mathbb{R}$ is continuously differentiable. If the segment joining the points $\mathbf{x}$ and $\mathbf{x}+\mathbf{h}$ lies in U, then there is a number $\theta \in (0,1)$ such that

$$f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) = (\nabla f(\mathbf{x}+\theta\mathbf{h}), \mathbf{h}).$$

In this paper, we will proceed the Mean Value Theorem in a different direction. If the surface is given explicitly as $f(x,y,z) = 0$, then the normal vector at a point (x,y,z) on the surface is given by the gradient vector $\nabla f(x,y,z)$. If a surface S is given explicitly as $z = f(x,y)$, then we write $F(x,y,z) = f(x,y) - z$ and we consider the surface $z = f(x,y)$ as the level surface of $F(x,y,z) = 0$. We can also describe a surface S in $\mathbb{R}^3$ in parametric form, as a vector-valued function of two parameters $w(u,v) = [x(u,v), y(u,v), z(u,v)]$. Throughout this paper, a parametric surface is assumed to be orientable. If the normal vector exists and $w_u \times w_v$ is not $\mathbf{0}$, then the surface S is called smooth. (There are no sharp 'corners'). In such case, the tangent plane to S at a point exists. In Theorems 1 and 2, we describe the Rolle's Theorems in $\mathbb{R}^3$. For Theorem 3 and Corollary 4, we describe the Mean Value Theorem in $\mathbb{R}^3$. In Section 3, we replace the intersecting plane P (with the given surface) in MVT by another smooth surface and obtain a similar result.

Analogous to a tangent plane in $\mathbb{R}^3$, for $n > 3$, the hyperplane is a linear equation consists of a set of points $\mathbf{x}$ in $\mathbb{R}^{n-1}$ satisfying

$$\overrightarrow{n}\cdot(\overrightarrow{\mathbf{x}} - \overrightarrow{\mathbf{x}_0}),$$

where $\mathbf{x}_0$ is a point on the hyperplane and $\overrightarrow{n}$ is a given direction, which is the normal direction to the hyperplane. The method of finding the appropriate tangent planes in $\mathbb{R}^3$ can be extended to finding appropriate $\mathbb{R}^{n-1}$ hyperplane in $\mathbb{R}^n$ regardless if a surface is given in implicit form of $f(x_1, x_2, ..., x_n) = 0$ or in parametric form of $w(u_1, u_2, ..., u_{n-1})$ (see [3]). The Theorem 5 generalizes the MVT in $\mathbb{R}^n$. The Theorem 6 links the generalized MVT with problems related to finding the extremum with several constraints. However, for demonstration purpose, we shall focus most cases in $\mathbb{R}^3$ in this paper unless otherwise is stated. We use the following notations: For a set $A \subset \mathbb{R}^n$, A^o denotes the interior of the set A, and $\overline{A}$ is the closure of the set A.

2 Mean Value Theorem in Higher Dimensions

The next two theorems describe versions of Rolle's Theorems in $\mathbb{R}^3$, one in function form, and the other is in parametric form.

Theorem 1 *Let f be a bounded function defined on $\mathbb{R}^2$. We assume a horizontal plane P of $z = k$, where k is a constant, intersects the surface $z = f(x,y)$ in union of finitely many space curves $C_i, i = 1, 2, ...n$, and the $xy-$projections of the intersection is a union of finitely region D_i, $i = 1, 2, ...n$. If f is differentiable over $\cup_{i=1}^{n}(D_i)^o$ and continuous on $\cup_{i=1}^{n}\overline{D_i}$, then there exists a point (x_0, y_0) in $\cup_{i=1}^{n}(D_i)^o$ such that f has a horizontal plane at $(x_0, y_0, f(x_0, y_0))$.*

Proof. Let the horizontal plane P intersect $z = f(x,y)$ at the surface $z = f(x,y)$ in union of finitely many space curves $C_i, i = 1, 2, ...n$, and the $xy-$projection of $P\cup(\cup_{i=1}^{n}C_i)$ is union of finitely region D_i, $i = 1, 2, ...n$.We proceed to show that either f is constant in $\cup_{i=1}^{n}D_i$, in which case, the tangent plane is horizontal at every point on $\cup_{i=1}^{n}D_i$, or f has a relative extremum at a point (x_0, y_0) in $(\cup_{i=1}^{n}D_i)^o$, in which case f has a horizontal plane at $(x_0, y_0, f(x_0, y_0))$. First we note that f is continuous and bounded on $\cup_{i=1}^{n}\overline{D_i}$, it assumes its maximum value M and minimum value m somewhere on $\cup_{i=1}^{n}\overline{D_i}$ by Extremum Value Theorem. If $M = m$,then f is constant on $\cup_{i=1}^{n}\overline{D_i}$, and the tangent plane is horizontal at every point on $\cup_{i=1}^{n}\overline{D_i}$. On the other hand, if $k = M$ or $k = m$, we are done. Suppose $M \neq m \neq k$, since $f(x,y) = k$ for all (x,y) on $\partial(\cup_{i=1}^{n}D_i)$, we know at least $f(x,y) \neq M$ or $f(x,y) \neq m$ for some $(x,y) \in (\cup_{i=1}^{n}D_i)^o$. Suppose $M > f(x,y) = k$, where $(x,y) \in \partial(\cup_{i=1}^{n}D_i)$. There exists an $(x_0, y_0) \in (\cup_{i=1}^{n}D_i)^o$ such that $f(x_0, y_0) = M$ and thus f has a relative maximum at (x_0, y_0), and hence f has a horizontal tangent plane at (x_0, y_0). Similar argument can be done for $m < f(x,y) = k$, where $(x,y) \in \partial(\cup_{i=1}^{n}D_i)$. ■

Theorem 2 *Let $w(u,v)$ be a bounded parametric surface in $\mathbb{R}^3$, and P be a horizontal plane which intersects $w(u,v)$ in union of finitely many smooth space curves $C_i, i = 1, 2, ...n$, and the $xy-$projections of the intersection is a union of finitely region D_i, $i = 1, 2, ...n$. If $w(u,v)$ is differentiable over $\cup_{i=1}^{n}(D_i)^o$ and continuous on $\cup_{i=1}^{n}\overline{D_i}$ for all $(u,v) \in \cup_{i=1}^{n}\overline{D_i}$. Then there exists $(u_0, v_0) \in \cup_{i=1}^{n}(D_i)^o$ such that the tangent plane at $(u_0, v_0, w(u_0, v_0))$ is parallel to the horizontal plane P.*

Proof. For simplicity, we assume the horizontal plane P intersects the bounded surface $w(u,v)$ at a smooth space curve C, and the $xy-$projection of the intersection enclosed a region D. If there is a $k \in \mathbb{R}$ such that $z = k$ intersects the surface $w(u,v)$ as a function, then we apply the Theorem 1, and there exists $(u_0, v_0) \in \cup_{i=1}^{n}(D_i)^o$ such that the tangent plane at $(u_0, v_0, w(u_0, v_0))$ is parallel to the horizontal plane P. Suppose there is no such $k \in \mathbb{R}$ such that the bounded surface becomes a function, then the surface $w(u,v)$ must be unbounded, which is a contraction. ■

The following theorem can be viewed as the Cauchy Mean Value Theorem (CMVT) for a parametric surface in $\mathbb{R}^3$.

Theorem 3 *Let $w(u,v)$ be a bounded parametric surface in $\mathbb{R}^3$, and P be the plane of the form $ax + by + cz = d$, which intersects $w(u,v)$ in union of finitely many space curves $C_i, i = 1, 2, ...n$, and the $xy-$projections of the intersection enclosed union of finitely many region D_i, $i = 1, 2, ...n$. If $w(u,v)$ is differentiable over $\cup_{i=1}^{n}(D_i)^o$ and continuous on $\cup_{i=1}^{n}\overline{D_i}$ for all $(u,v) \in \cup_{i=1}^{n}D_i$. Then there exists $(u_0, v_0) \in \cup_{i=1}^{n}(D_i)^o$ and $k \in \mathbb{R}$ such that the tangent plane at*

$(u_0, v_0, w(u_0, v_0))$ *is parallel to the plane* P*, and the followings are satisfied at the point* (u_0, v_0)*:*

$$\begin{aligned} z_u &= \frac{-ax_u - by_u}{c}, \\ z_v &= \frac{-ax_v - by_v}{c}, \\ x_u y_v - y_u x_v &= k. \end{aligned}$$

Proof. For simplicity, we show that the surface $w(u, v)$ intersects the plane, $ax + by + cz = d$, at a smooth space curve C, and we call the $xy-$projection of the intersection to be the region D. We consider the surface $w^*(u, v) = [x(u, v), y(u, v), z(u, v) - \dfrac{d - ax(u, v) - by(u, v)}{c}]$ for all $(u, v) \in D$. We note that the horizontal plane $z = 0$ intersects $w^*(u, v)$ for all $(u, v) \in D$. It follows from the Theorem 2 that there exists $(u_0, v_0) \in \cup_{i=1}^{n} (D_i)^o$ such that the tangent plane at $(u_0, v_0, w^*(u_0, v_0))$ is parallel to $z = 0$. In other words, $w_u^* \times w_v^*$ is parallel to $(0, 0, k)$ for some $k \in \mathbb{R}$.We write $w_u^* \times w_v^* = [w_1, w_2, w_3]$ and observe the followings:

$$\begin{aligned} w_1 &= y_u \left(z_v - \frac{-ax_v - by_v}{c} \right) - y_v \left(z_u - \frac{-ax_u - by_u}{c} \right), \\ w_2 &= x_v \left(z_u - \frac{-ax_u - by_u}{c} \right) - x_u \left(z_v - \frac{-ax_v - by_v}{c} \right), \\ w_3 &= x_u y_v - y_u x_v. \end{aligned}$$

Since $w_1 = w_2 = 0$, and $w_3 = k$ for some $(u_0, v_0) \in D$ and some $k \in \mathbb{R}$, we have

$$\begin{aligned} z_u &= \frac{-ax_u - by_u}{c}, \\ z_v &= \frac{-ax_v - by_v}{c}, \\ x_u y_v - y_u x_v &= k. \end{aligned}$$

■

Corollary 4 *Let* $f(x, y)$ *be differentiable over an open region* D *and continuous over* $\overline{D}$*. We assume a non-vertical plane* P *of* $ax + by + cz = d$ *intersects the surface* $z = f(x, y)$*,then we can find a point* X_0 *on* $z = f(x, y)$ *where the tangent plane at* X_0 *is parallel to the plane* P*. In other words, there exists* $k \in \mathbb{R}$ *such that the normal vector of the tangent plane at* X_0 *is* $(-\frac{a}{c}, -\frac{b}{c}, k)$*.*

Proof. We write $w(u, v) = [u, v, f(u, v)]$, the result follows directly from Theorem 3. ■

3 Extensions of MVT and Optimization Problems

We consider the smooth surface $g(x, y, z) = 0$ and a plane P satisfying the conditions stated in the Theorem 3. To find the desired point $X_0 = (x_0, y_0, z_0)$ on $g(x, y, z) = 0$ so that the tangent plane at X_0 parallel to P, $ax + by + cz = d$, is 'almost' equivalent to solving the following statement:

Find the extreme values of $f(x, y, z) = z - \dfrac{d - ax - by}{c}$ *subject to a constraint of the form* $g(x, y, z) = 0$. The differences can be seen in the following Theorems 5 and 6

We note that the necessary condition for finding the extreme value of l such that the surface $f(x, y, z) = l$ subject to the condition of $g(x, y, z) = 0$ is that $\nabla f(x, y, z)$ has to be a multiple of $\nabla g(x, y, z)$ at the point of tangency. It is natural one can replace $f(x, y, z) = z - \dfrac{d - ax - by}{c}$ by a more general smooth function. In the terminology of MVT, we may state that: Given two differentiable surfaces, $f(x, y, z) = 0$ and $g(x, y, z) = 0$. If the tangent plane for $g(x, y, z) = 0$ and $f(x, y, z) = k$ is the same at (x_0, y_0, z_0) for some $k \in \mathbb{R}$. Then there is a nonzero λ such that $\nabla f(x_0, y_0, z_0) = \nabla \lambda g(x_0, y_0, z_0)$. We certainly can extend this observation to the following:

Theorem 5 *We are given differentiable surfaces* $f(x_1, x_2, ..., x_n) = 0$,*and* $g_i(x_1, x_2, ..., x_n) = 0, i = 1, 2...p$. *Then* $(x_1^*, x_2^*, ..., x_n^*)$ *is a point on the surface of* $g_i(x_1, x_2, ..., x_n) = 0, i = 1, 2, ...p$, *such that the normal vector for the hyperplane of* $f(x_1, x_2, ..., x_n) = k$ *at* $(x_1^*, x_2^*, ..., x_n^*)$, *for some* $k \in \mathbb{R}$ *is a linear combination from a linearly independent set of vectors* $\{\nabla g_i(x_1^*, x_2^*, ..., x_n^*)\}_{i=1}^p$ ***if and only if*** *there are nonzero* $\lambda_i, i = 1, 2, ...p$ *such that* $\nabla f(x_1^*, x_2^*, ..., x_n^*) = \sum_{i=1}^p \lambda_i \nabla g_i(x_1^*, x_2^*, ..., x_n^*)$.

Alternatively, if we interpret the preceding problem as finding the extreme values of $f(x_1, x_2, ..., x_n)$ subject to the $p-$constraints $g_i(x_1, x_2, ..., x_n) = 0, i = 1, 2...p$. Then we may apply Lagrange Multipliers Method to solve the extreme value problem with several constraints. We state the following without proof, which can be found in many regular textbooks.

Theorem 6 *We assume that* f, g_i *are continuously differentiable:* $\mathbb{R}^n \to \mathbb{R}$, *with* $i = 1, 2, ...p$. *Suppose that we want to maximize or minimize a function of* n *variables* $f(\mathbf{x}) = f(x_1, x_2, ..., x_n)$ *for* $\mathbf{x} = (x_1, x_2, ..., x_n)$ *subject to* p *constraints* $g_1(\mathbf{x}) = c_1, g_2(\mathbf{x}) = c_2, ...,$ *and* $g_p(\mathbf{x}) = c_p$. *The necessary condition of finding the relative maximum or minimum of* $f(\mathbf{x})$ *subject to the constraints* $g_1(\mathbf{x}) = c_1, g_2(\mathbf{x}) = c_2, ...,$ *and* $g_p(\mathbf{x}) = c_p$ *that is not on the boundary of the region where* $f(\mathbf{x})$ *and* $g_i(\mathbf{x})$ *are defined can be found by solving the system*

$$\frac{\partial}{\partial x_i}\left(f(\mathbf{x}) + \sum_{j=1}^{p} \lambda_j g_j(\mathbf{x})\right) = 0,\ 1 \le i \le n, \tag{1}$$

$$g_j(\mathbf{x}) = c_j,\ 1 \le j \le p. \tag{2}$$

We write $\nabla f(\mathbf{x}) = \left(\dfrac{\partial}{\partial x_1} f(\mathbf{x}), \dfrac{\partial}{\partial x_2} f(\mathbf{x}), ..., \dfrac{\partial}{\partial x_n} f(\mathbf{x})\right)$. *If* $x = \mathbf{x}_0$ *is an extremum for above system, then*

$$\nabla f(\mathbf{x}_0) = \sum_{j=1}^{p} \lambda_j \nabla g_j(\mathbf{x}_0). \tag{3}$$

3.1 Examples

In this subsection, we demonstrate how Theorems 5 and 6 can be adopted to find the desired solutions computationally through the help of a CAS such as [6] or [7].

Example 7 *Consider the ellipsoid* $w(u,v) = [x(u,v), y(u,v), z(u,v)] = [\cos u \sin v, \sin u \sin v, 2\cos v]$, *where* $u \in [0, 2\pi]$ *and* $v \in [0, \pi]$, *and the plane* $P : 4x + 3y - z = 0$. *Find a point* X_0 *on the ellipsoid so that the tangent plane at* X_0 *is parallel to* P.

Method 1. (Lagrange) We may rewrite the parametric surface in rectangular form $x^2 + y^2 + \frac{z^2}{4} = 1$, and the problem can be stated as the following: *Find the extreme values of* $f(x,y,z) = z - 4x - 3y$ *subject to a constraint of* $x^2 + y^2 + \dfrac{z^2}{4} = 1$.

We set $L(x,y,z,\lambda) = z - 4x - 3y - \lambda(1 - x^2 - y^2 - \dfrac{z^2}{4})$, and set $\nabla L = 0$ to solve for x, y, z, and λ.With the help of CAS [6], we get

$$\lambda = \pm\frac{\sqrt{29}}{2}, x = \mp\frac{4}{\sqrt{29}}, y = \mp\frac{3}{\sqrt{29}}, \text{ and } z = \pm\frac{4}{\sqrt{29}}.$$

We choose

$$\{\lambda = 2.692582404, x = -.7427813528, y = -.5570860147, \text{ and } z = .7427813528\}$$

for demonstration and leave the other as an exercise. Thus, we get
$X_0 = [-0.7427813528, -0.5570860147, 0.7427813528]$.

Method 2. (Apply the Theorem 3) We follow the proof mentioned in Theorem 3 by considering the surface of

$$\begin{aligned} w^*(u,v) &= [[x(u,v), y(u,v), z(u,v) - 4x(u,v) - 3y(u,v)] \\ &= [\cos u \sin v, \sin u \sin v, 2\cos v - 4\cos u \sin v - 3\sin u \sin v. \end{aligned}$$

We find $w_u^* = [-\sin u \sin v, \cos u \sin v, 4\sin u \sin v - 3\cos u \sin v]$,
and $w_v^* = [\cos u \cos v, \sin u \cos v, -2\sin v - 4\cos u \cos v - 3\sin u \cos v]$. Therefore, if we write $w_u^* \times w_v^* = [a, b, c]$, we obtain

$$\begin{aligned} a &= \cos u \sin v\,(-2\sin v - 4\cos u \cos v - 3\sin u \cos v) \\ &\quad - (4\sin u \sin v - 3\cos u \sin v)\sin u \cos v \\ &= 4 + 2\tan v \cdot \sin u, \\ b &= (4\sin u \sin v - 3\cos u \sin v)\cos u \cos v + \\ &\quad \sin u \sin v\,(-2\sin v - 4\cos u \cos v - 3\sin u \cos v) \\ &= 3 + 2\tan v \cdot \cos u, \text{ and} \\ c &= -\sin^2 u \sin v \cos v - \cos^2 u \sin v \cos v \\ &= -\sin v \cos v. \end{aligned}$$

By setting $a = 0$, $b = 0$ and $c = k$, we solve u, v and k, and get the followings:

$$\{k = 0., u = u, v = 0.\},$$
$$\{k = .3448275862, u = .6435011088, v = -1.190289950\} \text{ and}$$
$$\{k = -.3448275862, u = -2.498091545, v = 1.190289950\}.$$

We substitute $\{u = -2.498091545, v = 1.190289950\}$ into $w^*(u, v)$ to obtain the point $X_0^* = [-.7427813528, -.5570860146, 5.385164807]$. (We note that $\{k = 0., u = u, v = 0.\}$ is not suitable and leave the other solution as an exercise). We plot the surface $w^*(u, v)$ and $z = 5.385164807$ in Figure 1. As expected, the surface $w^*(u, v)$ has a horizontal tangent at X_0^*.

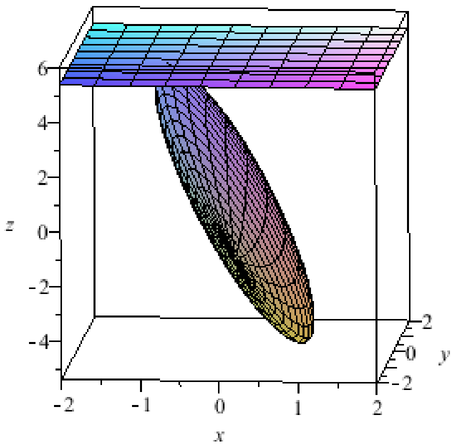


Figure 1. Rotated surface and the horizontal tangent plane

In addition, we note that the respective x and y values for X_0^* and X_0 (from Method 1) are identical. We show in Figure 2 below that the plane P_1 of $4(x + .7427813528) + 3(y + .5570860146) - (z - .7427813522) = 0$ is the tangent plane at X_0^* and is parallel to the given plane P of $4x + 3y - z = 0$.

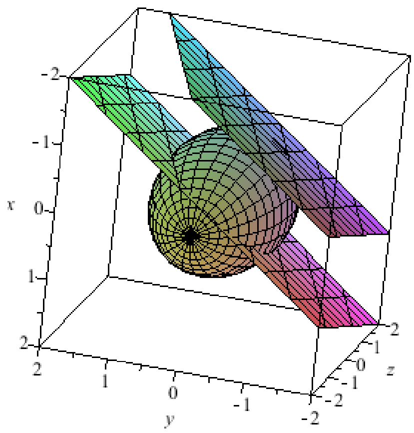


Figure 2. Surface with slanted plane and tangent plane

Discussions: For Example 7, we may generalize the problem as *finding the extreme values of* $f(x,y,z) = z - mx - ly$ *subject to a constraint of* $x^2+y^2+\frac{z^2}{n} = 1$. We define $L(x,y,z,\lambda) = z - mx - ly - \lambda\left(x^2+y^2+\frac{z^2}{n}-1\right)$, and set $\nabla L = 0$. We demonstrate solutions with the help of CAS [6] as follows::

1. For Method 1 we obtain

$$\lambda = \pm\frac{\sqrt{m^2+l^2+n}}{2}, x = -\frac{m}{2\lambda}, y = -\frac{l}{2\lambda}, z = \frac{n}{2\lambda} \text{ and } f(x,y,z) = 2\lambda.$$

We have two solutions for this problem, namely, $B = \left[\frac{-m}{\sqrt{m^2+l^2+n}}, \frac{-l}{\sqrt{m^2+l^2+n}}, \frac{n}{\sqrt{m^2+l^2+n}}\right]$ and $C = \left[\frac{m}{\sqrt{m^2+l^2+n}}, \frac{l}{\sqrt{m^2+l^2+n}}, \frac{-n}{\sqrt{m^2+l^2+n}}\right]$.

2. For Method 2, by applying the Theorem 3, by writing $w_u^* \times w_v^* = [a,b,c]$, we obtain

$$\begin{aligned} a &= m + \tan v \cdot \sin u \cdot \sqrt{n}, \\ b &= l + \tan v \cdot \cos u \cdot \sqrt{n}, \text{ and} \\ c &= -\sin v \cos v. \end{aligned}$$

By setting $(a,b,c) = (0,0,k)$,and with some algebraic simplifications and note that $u \in [0,2\pi]$ and $v \in [0,\pi]$, we obtain

$$\left\{u = \arctan\frac{l}{m}, v = \pm\arctan\sqrt{\frac{m^2+l^2}{n}} \text{ and } k = \pm\frac{\sqrt{n\,(m^2+l^2)}}{m^2+l^2+n}\right\}.$$

If we substitute $\left\{u = \arctan\frac{l}{m}, v = \arctan\sqrt{\frac{m^2+l^2}{n}}\right\}$ into $w^*(u,v)$, we obtain the point $B' = [x(u,v), y(u,v), z(u,v) - mx(u,v) - ly(u,v)]$. If we substitute $\left\{u = \arctan\frac{l}{m}, v = -\arctan\sqrt{\frac{m^2+l^2}{n}}\right\}$, we obtain the second solution at the point $C' = [x(u,v), y(u,v), z(u,v) - mx(u,v) - ly(u,v)]$.

3. For the video demonstration on this general case, please see [8].

In the next example, we replace the linear function $f(x,y,z)$ by a smooth function.

Example 8 *We consider two differentiable surfaces* $f(x,y,z) = z - 4x^2 - 3y^2$ *and* $g(x,y,z) = 1 - x^2 - y^2 - \frac{z^2}{4}$. *Then find a point* $X = (x_0,y_0,z_0)$ *on the surface of* $g(x,y,z) = 0$ *such that the tangent plane of* $g(x,y,z) = 0$ *at* X *is the same as the tangent plane of* $f(x,y,z) = k$ *at* X *for some* $k \in \mathbb{R}$.

We consider $L(x, y, z, \lambda) = f(x, y, z) + \lambda g(x, y, z)$ and set $\nabla L = 0$ to solve for x, y, z, and λ. With the help of Maple (see [7]), we obtain

$$\{\lambda = -4., x = .9682458365, y = 0., z = -.5000000000\},$$
$$\{\lambda = -3., x = 0., y = .9428090414, z = -.6666666667\},$$
$$\{\lambda = 1., x = 0., y = 0., z = 2.\},$$
$$\{\lambda = -1., x = 0., y = 0., z = -2.\}.$$

We consider the following two cases and leave the others to readers to explore:

Case 1. $\{\lambda = -4., x = .9682458365, y = 0., z = -.5000000000\}$ We note that $f(.9682458365, 0, -0.5) = -4.25$, we plot the graphs of $f(x, y, z) = -4.25$ and $g(x, y, z) = 0$ in Figure 3(a) with [7], it is clear that these two surfaces are tangent to each other at the desired point of $x = .9682458365, y = 0$, and $z = -.5000000000$.

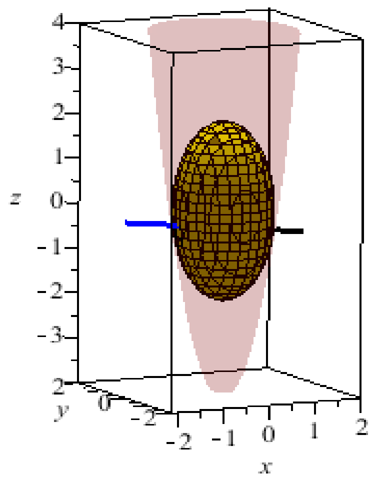


Figure 3(a). When replacing the hyperplane with a surface in MVT-Case 1

Case 2. $\{\lambda = 1, x = 0, y = 0., z = 2\}$ We note that $f(0, 0, 2) = 2$, we plot the graphs of $f(x, y, z) = 2$ and $g(x, y, z) = 0$ in Figure 3(b) with [7], it is clear that these two surfaces are tangent to each other at the desired point of $x = 0, y = 0$, and $z = 2$.

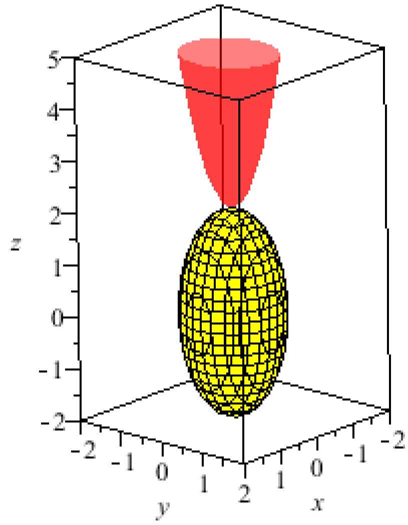


Figure 3(b). When replacing the hyperplane with a surface in MVT-Case 2

Exercise. Use the Method 2 mentioned in Example 7 to obtain the same results as described above.

Example 9 *We consider differentiable surfaces* $f(x, y, z) = (x + 1)^2 + y^2 + (z-3)^2 - 1, g_1(x, y, z) = \frac{1}{9}x^2 + y^2 + z^2 - 1 = 0, g_2(x, y, z) = \frac{1}{4}x^2 + \frac{1}{4}y^2 + z^2 - 1 = 0,$ *and* $g_3(x, y, z) = x^2 + y^2 + z^2 - 2 = 0$. *If possible, find a point* (x_0, y_0, z_0) *on the surfaces of* $g_i(x, y, z) = 0, i = 1, 2,$ *and* 3, *such that the normal vector for the tangent plane of* $f(x, y, z) = k$ *at* (x_0, y_0, z_0), *for some* $k \in \mathbb{R}$ *is a linear combinations of* $\nabla g_i(x_0, y_0, z_0)$, *where* $i = 1, 2,$*and* 3.

We first note that if such solution exists, the surface $f(x, y, z)$ will touch **exactly at the point** where three surfaces $g_i(x, y, z) = 0$ intersect, $i = 1, 2$, and 3. We consider

$$L(x, y, z, \lambda_1, \lambda_2, \lambda_3) = f(x, y, z) + \lambda_1 g_1(x, y, z) + \lambda_2 g_2(x, y, z) + \lambda_3 g_3(x, y, z),$$

and set $\nabla L = 0$ to solve for $x, y, z, \lambda_1, \lambda_2$, and λ_3. This amounts to solving for the following two parts:

Part 1. Solve x, y and z from $g_i(x, y, z) = 0$, for $i = 1, 2$, and 3. We get the following *eight solutions*:

$$\left\{ x = \pm\frac{3\sqrt{2}}{4}, y = \pm\sqrt{\frac{5}{24}}, z = \pm\sqrt{\frac{2}{3}} \right\}.$$

Part 2. Solve for λ_1, λ_2, and λ_3 in terms of x, y and z, with the help of [6] or [7], we obtain the following:

$$\left\{ \lambda_1 = \frac{9}{8x}, \lambda_2 = \frac{4}{z}, \lambda_3 = -\frac{8xz + 8x + 9z}{8xz} \right\}.$$

We use the following two cases for demonstrations and leave the rest to readers to explore analogously.

Case 1. $\left\{x_0 = -\frac{3\sqrt{2}}{4}, y_0 = \sqrt{\frac{5}{24}}, z_0 = -\sqrt{\frac{2}{3}}, \lambda_1 = -1.060660172, \lambda_2 = -4.898979486, \lambda_3 = 1.2854050\right\}$

We note that $f(x,y,z) = f(x_0,y_0,z_0) = 13.77765914$ is tangent to all of the surfaces $g_i(x,y,z) = 0, i = 1,2,3,$ at (x_0,y_0,z_0). Furthermore, it follows from Theorem 5 that $\nabla f(x_0,y_0,z_0)$ can be written as a linear combinations of the $\nabla g_i(x,y,z) = 0, i = 1,2,3$. We demonstrate this by using the following Figures 4(a) and 4(b) with the help of [6]. The $g_1(x,y,z) = 0$ is shown in yellow, $g_2(x,y,z) = 0$ is shown in blue, $g_3(x,y,z) = 0$ is shown in green, and $f(x,y,z) = f(x_0,y_0,z_0) = 13.77765914$ is shown in red.

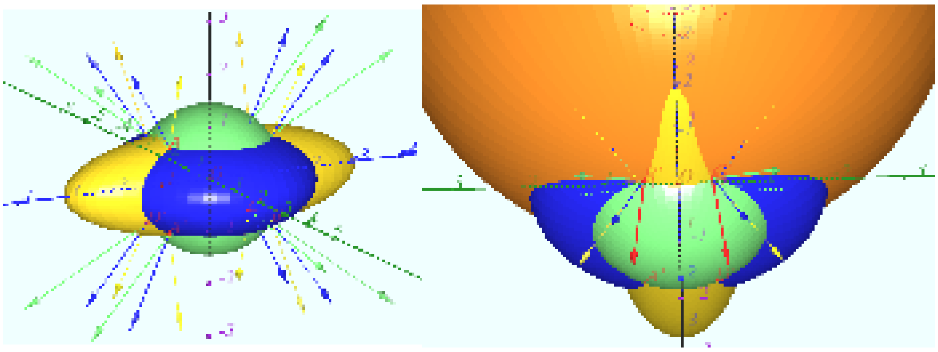


Figure 4(a). Case 1 of MVT with intersecting surfaces.

Figure 4(b). Case 1 of MVT with respective normal vectors

Case 2. $\left\{x_0 = \frac{3\sqrt{2}}{4}, y_0 = \sqrt{\frac{5}{24}}, z_0 = \sqrt{\frac{2}{3}}, \lambda_1 = 1.060660172, \lambda_2 = 4.898979486, \lambda_3 = -3.285405043\right\}$

We note that $f(x,y,z) = f(x_0,y_0,z_0) = 8.222340858$ is tangent to all of the surfaces $g_i(x,y,z) = 0, i = 1,2,3,$ at (x_0,y_0,z_0). It follows from Theorem 5 that $\nabla f(x_0,y_0,z_0)$ can be written as a linear combinations of the $\nabla g_i(x,y,z) = 0, i = 1,2,3$. We demonstrate this by using the following Figures 5(a) and 5(b) with the help of [6]. The $g_1(x,y,z) = 0$ is shown in yellow, $g_2(x,y,z) = 0$ is shown in blue, $g_3(x,y,z) = 0$ is shown in green, and $f(x,y,z) = f(x_0,y_0,z_0) = 8.222340858$ is shown in red.

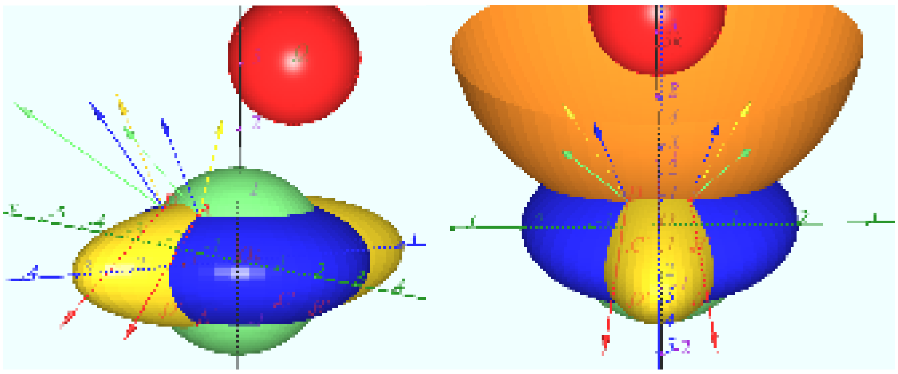


Figure 5(a). Case 2 of MVT with intersecting surfaces.

Figure 5(b). Case 2 of MVT with respective normal vectors

Remarks:

1. As we have mentioned if such a solution (x_0, y_0, z_0) exists, the surface $f(x, y, z) = f(x_0, y_0, z_0)$ is tangent to all of the surfaces $g_i(x, y, z) = 0, i = 1, 2, 3$, at (x_0, y_0, z_0). Furthermore, the $\nabla f(x_0, y_0, z_0)$ can be written as a linear combinations of the $\nabla g_i(x, y, z) = 0, i = 1, 2, 3$.

2. We see from Example 9 that since there are eight intersections for the surfaces of $g_i(x, y, z) = 0, i = 1, 2, 3$, all gradient vectors at any particular intersection (x_0, y_0, z_0), $\nabla g_1(x_0, y_0, z_0)$, $\nabla g_2(x_0, y_0, z_0)$, and $\nabla g_3(x_0, y_0, z_0)$ form a linearly independent set in $\mathbb{R}^3$. Furthermore, $\nabla f(x_0, y_0, z_0)$ can be written as linear combinations of the gradient vectors $\{\nabla g_i(x_0, y_0, z_0)\}_{i=1}^3$; of course, we assume all gradient vectors here are not zero or parallel. We demonstrate this by using Figure 5(c) below:

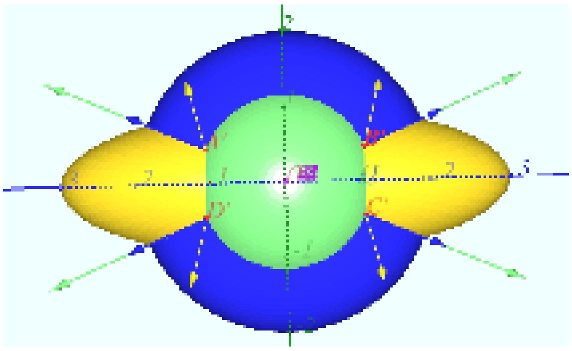


Figure 5(c). Gradient vectors are linearly independent

Corollary 10 *We assume that f, g_i are continuously differentiable: $\mathbb{R}^3 \to \mathbb{R}$, with $i = 1, 2$, and 3. Suppose $f(\mathbf{x})$ has a extreme values subject to three constraints $g_1(\mathbf{x}) = c_1, g_2(\mathbf{x}) = c_2$,and $g_3(\mathbf{x}) = c_3$, where $\mathbf{x} = (x_1, x_2, x_3)$. Furthermore, these three constraints, $g_1(\mathbf{x}) = c_1, g_2(\mathbf{x}) = c_2$,and $g_3(\mathbf{x}) = c_3$, intersect at a space curve C in $\mathbb{R}^3$.Then $\nabla f(\mathbf{x}^*)$ and $\{\nabla g_j(\mathbf{x}^*)\}_{i=1}^3$ are coplanar for all $\mathbf{x}^* \in C$.*

Proof. Suppose f has an extreme value at $\mathbf{x}^* \in C$, we note that $\nabla f(\mathbf{x}^*)$ is orthogonal to C at $\mathbf{x}^*$. Since $\{\nabla g_j(\mathbf{x})\}_{i=1}^3$ is orthogonal to $\{g_j(\mathbf{x}) = c_i\}_{i=1}^3$ respectively, we see $\{\nabla g_j(\mathbf{x}^*)\}_{i=1}^3$ is orthogonal to C. This implies that $\nabla f(\mathbf{x}^*)$ is a linear combination of $\{\nabla g_j(\mathbf{x})\}_{i=1}^3$, since dimension of $\mathbb{R}^3$ is 3, we see that $\{\nabla g_j(\mathbf{x}^*)\}_{i=1}^3$ are coplanar and thus $\nabla f(\mathbf{x}^*)$ and $\{\nabla g_j(\mathbf{x}^*)\}_{i=1}^3$ are coplanar for all $\mathbf{x}^* \in C$. ■

In the next example, we shall see how Corollary10 works.

Example 11 *We consider the differentiable function $f(x, y, z) = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$, which (x_0, y_0, z_0) is denoted by P.Next, we conside rthe ellipsoid $g_1(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} - 1 = 0$, and we denote O to be the origin and $a = AO$, $b = BO$. Before we construct the surfaces $g_2(x, y, z) = 0$ and $g_3(x, y, z) = 0$, we construct the horizontal circle $L : x^2 + y^2 = r^2$ with radius $r = \frac{\sqrt{3}a}{2}$ and center $C = (0, 0, \frac{b}{2})$. Now, we construct $g_2(x, y, z) = 0$ to be the torus with its horizontal cross section to be the circle L and denote the center of such torus ring as D, which is shown in Figure 5(d) below, with minor radius r and major radius CD. Finally, we construct the cone $g_3(x, y, z) = (x - x_1 t)^2 + (y - y_1 t)^2 + (z - z_1 t)^2 = r^2(1 - t)^2$, where $t = \frac{2z - b}{2z_1 - b}$ and (x_1, y_1, z_1) is the vertex E of the cone, which we show in Figure 5(d). We want to find a* *point* $X(x^*, y^*, z^*)$ *on all surfaces of* $g_i(x, y, z) = 0$, $i = 1, 2,$ *and* 3 *such that the normal vector for the tangent plane of* $f(x, y, z) = k^2$ *at* $X(x^*, y^*, z^*)$, *for some* $k \in \mathbb{R}$, *is a linear combination of* $\nabla g_i(x, y, z)$, *where* $i = 1, 2$ *and* 3.

We note the followings:

$$\nabla g_1(x^*, y^*, z^*) = (x^*, y^*, \frac{a^2}{b}),$$
$$\nabla g_1(x^*, y^*, z^*) = (x^*, y^*, 0) \text{ and}$$
$$\nabla g_1(x^*, y^*, z^*) = \left(x^*, y^*, \frac{r^2 - x^*x_1 - y^*y_1}{z_1 - 0.5b}\right),$$

where the normal vector are being normalized such that their x component is x^*. We observe all normal vectors are coplanar in this case and lie in a vertical plane Π which contains the $z-axis$ and the point X. Furthermore, we see $\nabla f(x^*, y^*, z^*) = 2(P-X)$, therefore, the surface $f(x, y, z) = k^2$ contains the point X if $k = |P - X|$. We discuss the following two possibilities:

1. Let $P \in \Pi$ but P is not on the $z - axis$: The normal vectors $\nabla g_i(x^*, y^*, z^*), i = 1, 2,$ and 3, and $\nabla f(x^*, y^*, z^*)$ belongs to the same plane and the normal vector for the tangent plane of $f(x, y, z) = k^2$ at $X(x^*, y^*, z^*)$ with $k = |P - X|$ is a linear combinations of $\nabla g_i(x^*, y^*, z^*), i = 1, 2,$ and 3. **There are two such points X belonging to the circle L for arbitrary P.**

2. Let P be on the $z - axis$, then $P \in \Pi$: The normal vectors $\nabla g_i(x^*, y^*, z^*), i = 1, 2,$ and 3, and $\nabla f(x^*, y^*, z^*)$ belong to the same plane and the normal vector for the tangent plane of $f(x, y, z) = k^2$at $X(x^*, y^*, z^*)$ with $k = |P - X|$ is a linear combinations of $\nabla g_i(x^*, y^*, z^*), i = 1, 2,$ and 3. For a video demonstration on this problem, we refer reader to see [9].

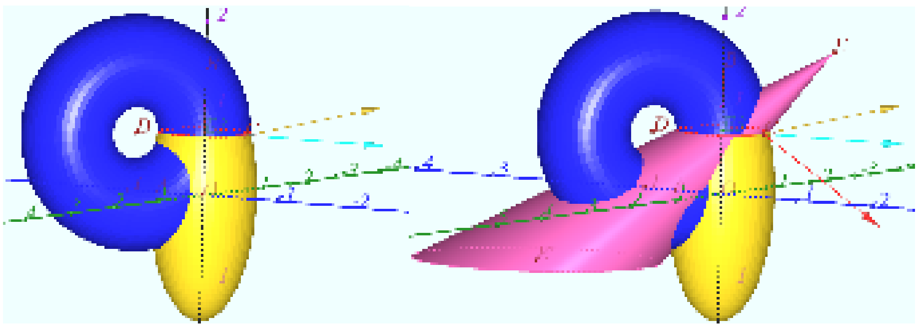


Figure 5(d): The ellipsoid and the torus.

Figure 5(e): The ellipsoid, torus and the cone.

We now consider an application of finding two parallel tangent planes at two points of two respective non-intersecting smooth surfaces. Given two non-intersecting smooth surfaces

$f(x, y, z) = 0$ and $g(x, y, z) = 0$, where f and g are continuously differentiable functions in their respective closed and bounded domains. Our task is find respective points on $f(x, y, z) = 0$ and $g(x, y, z) = 0$ such that the tangent planes at these respective points are parallel to each other. It follows from [5] that if such points exist for $f(x, y, z) = 0$ and $g(x, y, z) = 0$ respectively, the distance between these two points possibly produces an extreme value between these two points. It is clear that finding extreme value for the 'square distance' is a nice application of MVT in higher dimensions and Lagrange Multipliers.

We describe the extreme values for the square distance as follows: Let $f(x, y, z) = 0$ and $g(x, y, z) = 0$ be two non-intersecting surfaces, and we want to find the relative extremum squared distance between these two convex surfaces $f(x, y, z) = 0$ and $g(x, y, z) = 0$. If we write $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3)$, it follows from [5] that we want to minimize or maximize the squared distance $|\mathbf{x} - \mathbf{y}|^2$, which is subject to both $f(\mathbf{x}) = 0$ and $g(\mathbf{y}) = 0$. If we write

$$L(\mathbf{x}, \mathbf{y}, \lambda_1, \lambda_2) = |\mathbf{x} - \mathbf{y}|^2 + \lambda_1 f(\mathbf{x}) + \lambda_2 g(\mathbf{y}) \tag{4}$$

or

$$\begin{aligned} L(x_1, x_2, x_3, y_1, y_2, y_3, \lambda_1, \lambda_2) = (x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 + \\ \lambda_1 f(x_1, x_2, x_3) + \lambda_2 g(y_1, y_2, y_3); \end{aligned} \tag{5}$$

Then it follows from the Lagrange Multipliers Method that the necessary condition to achieve the critical distance is to set

$$\nabla L = 0, \tag{6}$$

and solve $x_1, x_2, x_3, y_1, y_2, y_3, \lambda_1$, and λ_2. The next example reminds us that the Lagrange Multipliers Method only provides a necessary but not a sufficient condition for finding the extremum value for the square distance function.

Example 12 *We consider the ellipsoid S_1 of the form $F(x, y, z) = X^T AX - r^2 = 0$, where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$, and $r = \sqrt{8}$. We see*

$$X^T AX - r^2 = x\,(3x + y + z) + y\,(x + 3y + z) + z\,(x + y + 5z) - 8 = 0.$$

We also consider the surface S_2 of the form $G(x, y, z) = \frac{1}{2}\left(\frac{1}{2}(x + 4)^2 + \frac{1}{2}(y + 5)^2 + \frac{1}{2}z^2 - 2\right)^2 + 2x + 18 + 2y - z = 0$. We want to find the shortest square distance between S_1 and S_2. In other words, we want to find the minimum of $|\mathbf{x} - \mathbf{y}|^2$, where $\mathbf{x} \in S_1$ and $\mathbf{y} \in S_2$. It is easy to see that we want

$$\begin{aligned} &\textit{to minimize } |\mathbf{x} - \mathbf{y}|^2 \\ &\textit{subject to} \\ &\mathbf{x} \in S_1 \textit{ and } \mathbf{y} \in S_2. \end{aligned}$$

If we set $L(\mathbf{x}, \mathbf{y}, \lambda_1, \lambda_2) = |\mathbf{x} - \mathbf{y}|^2 + \lambda_1 F(\mathbf{x}) + \lambda_2 G(\mathbf{y})$, we need to set $\nabla L = 0$ to solve for $\mathbf{x}, \mathbf{y}, \lambda_1$, and λ_2. With the help of Maple, we show the following two cases:

Case 1. *When* $\mathbf{x} = \begin{bmatrix} .8185635772 \\ 1.258751546 \\ -.5035606230 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} -6.282838309 \\ -7.789748993 \\ 0.4706522543 \end{bmatrix}, \lambda_1 = -2.211667277$ *and* $\lambda_{2.} = -1.667226467$. *We see* $|\mathbf{x} - \mathbf{y}|^2 = 133.2543615$, *we demonstrate the surfaces and the vector connecting* S_1 *and* S_2 *in Figure 6:*

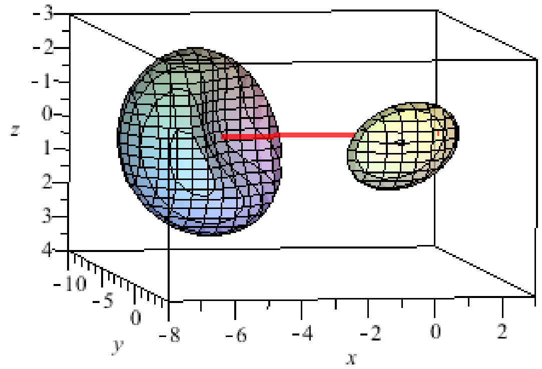


Figure 6. Minimum squared distance 1

Case 2. *When* $\mathbf{x} = \begin{bmatrix} -.9938820385 \\ -1.065237505 \\ .6812516238 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} -3.938343832 \\ -4.134551597 \\ 1.859811273 \end{bmatrix}, \lambda_1 = -1.522033890$ *and* $\lambda_{2.} = 5.068309708$. *We see* $|\mathbf{x} - \mathbf{y}|^2 = 19.47954710$, *we demonstrate the surfaces and the vector connecting* S_1 *and* S_2 *in Figure 7. It can be shown that the squared distance in this case produce the minimum value between two surfaces.*

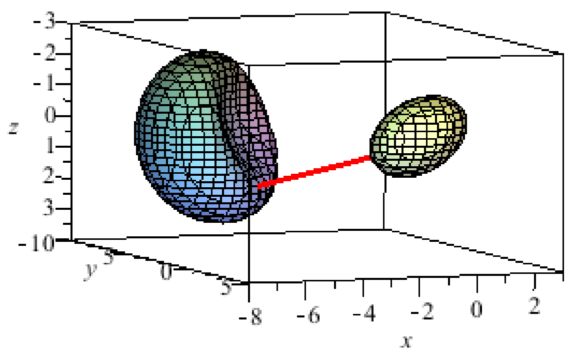
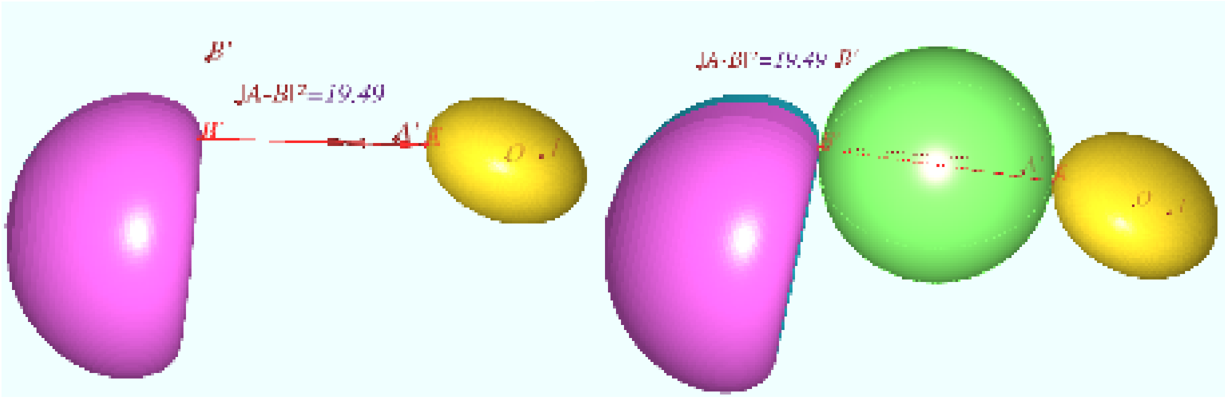


Figure 7. Minimum squared distance 2

The following Figures 8(a) and (b) created by [6] by give nice demonstrations how the respective *points in S_1 and S_2 produce the minimum squared distance:*

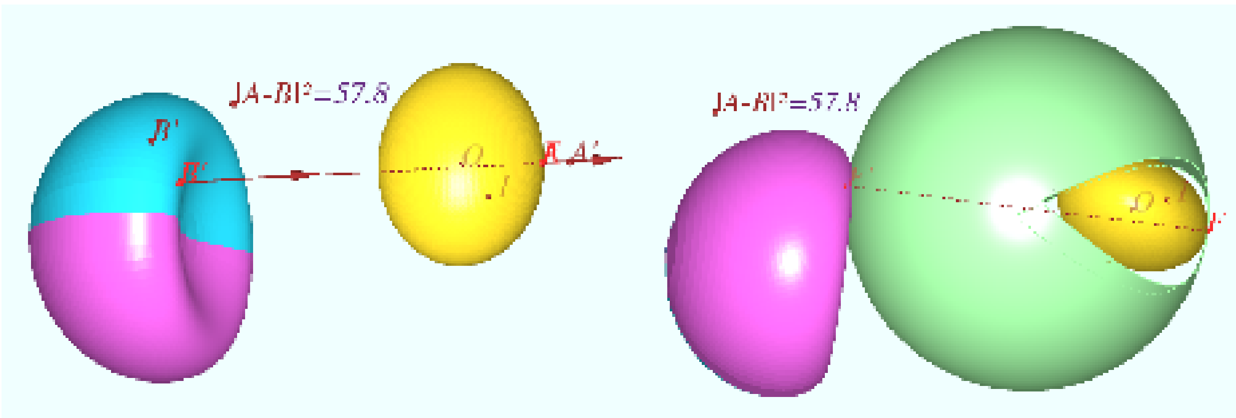


Figures 8(a) and 8(b) Minimum squared distance between S_1 and S_2

For a sufficient and necessary condition of finding the minimal distance between two non-convex surfaces, we refer readers to [2]. We refer readers to [5] for exploring more interesting problems in finding the extreme values of the total square distances among multiple surfaces.

Exercise. Explore that the following respective points between S_1 and S_2 produce neither maximum nor minimum square distances:

Case (i) $\mathbf{x} = \begin{bmatrix} -.8826421485 \\ -1.210622771 \\ 0.3862052730 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} -6.368158748 \\ -7.732409496 \\ 0.1299051204 \end{bmatrix}$, $\lambda_1 = 1.579773400$ and $\lambda_{2.} = -1.251743854$. We refer to the following Figures 9(a) and 9(b), created by [6] which show the respective points in S_1 and S_2 produce neither maximum nor minimum squared distance.



Figures 9(a) and 9(b) Neither maximum nor minimum squared distance.

Case (ii) $\mathbf{x} = \begin{bmatrix} 1.015161164 \\ 1.020326713 \\ -.7363979552 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} -4.052317116 \\ -4.062875848 \\ 1.769633735 \end{bmatrix}$, $\lambda_1 = -1.522033890$ and $\lambda_{2.} = 5.068309708$.

Discussions: For Example 12, we may generalize the problem to find the minimum squared distance between two disjoint surfaces determined by $f(x, y, z) = z - mx^2 - ny^2$ and $g(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$. For detailed demonstration, we refer readers to the video clip [9].

4 Conclusion

We have seen how we can extend the MVT and CMVT to higher dimensions, and interpret them nicely when we have multiple surfaces. Furthermore, we make a connection between MVT and CMVT with optimization problems with constraints. We give several examples to demonstrate that not only we believe the existence of a solution but also implement CAS to show how we can find desired solutions. We believe the contents in this paper is accessible to undergraduate students who have studied Multi-variable Calculus and Linear Algebra. We believe it is important to integrate the concepts between these two fields, which is essential before students study more deeper concepts in Differential Geometry.

References

[1] P. M. Fitzpatrick, '*Advanced Calculus*', Thomson Brooks/Cole, second edition, ISBN 0-534-37603-7, 2006.

[2] Y. Gao & W.-C. Yang, '*Complete Solutions to Minimal Distance Problem between Two Nonconvex Surfaces*', Journal of Optimization by Taylor & Francis, January 2008, pp 705-714.

[3] A. J. Hanson, '*Geometry for N-Dimensional Graphics*', Graphics Gems IV, edited by Paul S. Heckbert, 1994, Academic Press, ISBN 0-12-336156-7.

[4] W.-C. Yang, '*Revisit Mean Value, Cauchy Mean Value and Lagrange Remainder Theorems'*, Electronic Journal of Mathematics and Technology (eJMT), ISSN 1933-2823, Issue 2, Vol. 1, June, 2007.

[5] W.-C. Yang, '*Some Geometric Interpretations of The Total Distances Among Curves and Surfaces'*, Electronic Journal of Mathematics and Technology (eJMT), ISSN 1933-2823, Issue 1, Vol. 3, June, 2009.

5 Software Packages and Supplemental Electronic Materials

[6] GInMa Software, http://deoma-cmd.ru/en/Products/Geometry/GInMA.aspx.

[7] A product of Maplesoft, http://www.maplesoft.com/.

[8] Nosylia S, Shelomovskyi V., General case for Example 7, see video clip at http://www.youtube.com/watch?v=16hPQMATGJk.

[9] Nosylia S, Shelomovskyi V., General case for Example 10, see video clip at http://www.youtube.com/watch?v=v1k6ke6_L5k.